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Geometrical Aspects and Generalizations of Newton–Cartan Mechanics

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Cartan's geometrical approach to Newtonian mechanics is discussed and generalized. The approach is geometrical and use is made of the theory of holonomy groups. The concept of a degenerate metric connection is formulated and discussed.

1. INTRODUCTION

One of the most provocative aspects of classical Newtonian mechanics is the assumption of absolute space. This carries with it the ability to distinguish between 'true' and 'inertial' forces and leads to the idea of preferred 'inertial' frames in which 'inertial' forces are absent. Of course Newtonian mechanics can be written in a 'descriptively' covariant way by involving, for example, a Lagrangian formulation. The latter theory, however, contains the metric tensor components, which, though present in the equations of motion, do not themselves satisfy field equations. They are *absolute* variables in the sense of Anderson (1967) and Trautman (1965) and are merely a restatement of the original Euclidean nature of absolute space. Similar remarks apply to the way one writes Maxwell's equations in special relativity in covariant form by essentially allowing arbitrary coordinate systems and using some minimal coupling device. In each of these examples the mathematical model is a manifold carrying a metric and one has achieved a descriptive rather than a dynamical geometrization of the theory.

In general relativity, however, the principle of covariance is satisfied in a much stronger way. Einstein's field equations are written in a covariant

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form without the need to introduce any *absolute* variables. They contain only *dynamical* variables (i.e., the space-time metric). In this sense general relativity can claim to be a fully (i.e., dynamically) rather than a merely descriptively geometrized theory.

The descriptive geometrization of a theory can nevertheless be extremely instructive and illuminating and one often finds that certain features of the theory which are awkward in an analytical approach fit rather more elegantly into a geometrical framework. Cartan's approach to classical Newtonian mechanics (Cartan, 1923, 1924; Trautman, 1965; Misner *et al.*, 1973) is an example of a descriptive geometrization with a rather instructive geometrical content. It will be explored further in this paper with particular reference to the holonomy structure and some generalizations will be suggested. Historically, of course, Cartan's work is later than Einstein's general relativity and alternative geometrizations of classical mechanics had been developed before Einstein which were mainly due to the work of Euler, Lagrange, Jacobi, Hamilton, and others [a very readable account of this development can be found in Lanczos (1949)].

Cartan's treatment, however, differs from most of the other treatments in that it is based on a connection rather than a metric and links space and time together geometrically. Further, Cartan's connection is not a metric connection and so the techniques for handling it are a little different from those normally encountered in metric theories.

It should be noted that Cartan's geometrized theory and its relationship to Einstein's general relativity has been studied by a number of authors. In particular see Künzle (1972), Duval and Künzle (1977), and Dixon (1975) and the references contained in the reviews by Havas (1964) and Duval *et al.* (1985).

2. NEWTON–CARTAN MECHANICS

Let M be the manifold $\mathbf{R}^4$ with the standard global coordinate system denoted by (x^0, x^1, x^2, x^3). Here x^0 is to be regarded as *Newtonian absolute time* and the hypersurfaces of constant x^0 inherit a natural global coordinate system and hence a natural three-dimensional manifold structure from this coordinate system on M. Such hypersurfaces are referred to as *space slices* and are diffeomorphic to $\mathbf{R}^3$. It is supposed that on M a global type (2, 0) tensor h is defined whose components in the original coordinate system x^a above are constant and given by $h^{ab} = \operatorname{diag}(0, 1, 1, 1)$. Latin indices will take the values 0, 1, 2, 3 and Greek indices will take the values 1, 2, 3. The tensor h has rank 3 at each $p \in M$ and so is *not* a metric for M. The tensor h does, however, give rise to a unique tensor $\tilde{h}$ on each space slice which is a flat Euclidean metric with constant components $\tilde{h}^{\alpha\beta} = \mathrm{diag}(1, 1, 1)$ in the standard chart.

It is also assumed that M admits a smooth symmetric connection Γ defined by its components Γ^a_{bc} in the coordinates x^a which are

$$\Gamma^{\alpha}_{00} = -F^{\alpha} \tag{1}$$

All other components are zero and F^{α} are three given smooth, real-valued functions on M. Now it easily follows that any curve in M of the form $x^{\alpha} = a^{\alpha}\lambda + b^{\alpha}$, $x^0 = c$ (where $a^{\alpha}, b^{\alpha}, c \in \mathbf{R}$) is a geodesic of Γ (with affine parameter λ) which lies in the space slice $x^0 = c$. Since a vector $X \in T_pM$ is tangent to the space slice through p if and only if $X^0 = 0$, it now follows that any geodesic of M initially tangent to a space slice remains in the space slice and hence that each space slice is a *totally geodesic* submanifold of M. Because Γ is symmetric, it follows that each space slice is then an autoparallel submanifold of M (Kobayashi and Nomizu, 1963/1969, Vol. II), that is, each $X \in T_pM$ which is tangent to a space slice at p remains tangent to this space slice under parallel transport along any curve lying in this space slice. It follows that each space slice inherits a natural induced connection from the connection Γ on M. This natural induced connection is symmetric (since Γ is) and is then easily seen to be the Levi-Civita connection associated with $\tilde{h}$ on each space slice.

Further remarks can be made about the geometry of the connection Γ on M. First, the tensor h defined earlier is easily confirmed to be covariantly constant, i.e., $h^{ab}{}_{;c} = 0$, where a semicolon denotes the covariant derivative arising from Γ. Second, the globally defined 1-form $t \equiv dx^0$ [with components $t_a = (1, 0, 0, 0)$ in the above coordinate system] and the three globally defined vector fields

$$X = \frac{\partial}{\partial x^1} \qquad Y = \frac{\partial}{\partial x^2} \qquad Z = \frac{\partial}{\partial x^3}$$

[with respective components (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1) in the above coordinate system] are also covariantly constant. Third, the connection Γ gives rise to a curvature tensor $\mathcal{R}$ with components $R^a{}_{bcd}$ which in the above coordinate system satisfy

$$R^{\alpha}{}_{0\beta 0} = -R^{\alpha}{}_{00\beta} = -F^{\alpha}{}_{,\beta} \tag{2}$$

with all other components zero and where a comma denotes a partial derivative. Thus, apart from the special case when the functions F^{α} are each independent of the 'space' coordinates x^{α}, the connection Γ is *not flat*. Further, the equations $X^a{}_{;b} = Y^a{}_{;b} = Z^a{}_{;b} = 0$ and the Ricci identity show that

$$R^a{}_{bcd}X^b = R^a{}_{bcd}Y^b = R^a{}_{bcd}Z^b = 0 \tag{3}$$

This implies that *if $\mathcal{R}$ is not identically zero, then* Γ *is not a metric connection.* For if Γ is compatible with some metric, say g_{ab}, then the tensor $H_{abcd} = g_{ae}R^a{}_{bcd}$ would, in the usual 6×6 notation familiar from general relativity theory, admit six independent solutions for F^{ab} of the equation $F^{ab}H_{abcd} = 0$ [from (3)] and would thus be zero. This contradicts the assumption that $\mathcal{R}$ is not identically zero. Conversely, if $\mathcal{R}$ is identically zero, then it follows that, since $M = \mathbf{R}^4$ is simply connected, Γ is a metric connection. Hence it has been shown that Γ *is metric if and only if its associated curvature tensor is identically zero.*

Consider an affinely parametrized geodesic $x^a(\lambda)$ of M which is not tangent to the space slice at its initial point p. The previous remarks then show that the geodesic is never tangent to the space slice and (1) shows that its equation in the above coordinate system can be written in the form

$$\frac{d^2x^\alpha}{d\lambda^2} = F^\alpha\left(\frac{dx^0}{d\lambda}\right)^2, \qquad \frac{d^2x^0}{d\lambda^2} = 0 \tag{4}$$

This equation represents the central idea of Cartan's theory, since it shows that the absolute time function x^0 is an affine parameter for these geodesics of Γ and that these geodesics can then be regarded as the paths of particles moving under the influence of a Newtonian force represented in this coordinate system by the functions F^α.

If one performs a coordinate transformation $x'^0 = x^0, x'^\alpha = x^\alpha + f^\alpha(x^0)$, then the geodesic equation (4) becomes

$$\frac{d^2x'^\alpha}{d\lambda^2} = (F^\alpha + \ddot{f}^\alpha)\left(\frac{dx'^0}{d\lambda}\right)^2, \qquad \frac{d^2x'^0}{d\lambda^2} = 0 \tag{5}$$

where a dot represents differentiation with respect to x^0. From the physical viewpoint, if one regards the original coordinates as *inertial* [in the usual sense that any observer with the world line given by $x^\alpha = c^\alpha$ ($c^\alpha \in \mathbf{R}$) is an *inertial observer*], then (5) simply displays the extra inertial (accelerative) forces. This equation also shows that if Γ is flat [and hence $F^\alpha = F^\alpha(x^0)$ from (2)], then the force may be 'transformed away' by such a transformation if f^α is chosen appropriately. It can thus be seen that the flatness condition for Γ, the condition that Γ is metric, and the ability to 'transform the force away' in the above sense are equivalent conditions.

A connection Γ as described above will be referred to as a *standard Cartan connection for* $\mathbf{R}^4$. Usually F^α is given in the form $F^\alpha = \phi_{,\alpha}$, where ϕ is a real-valued function on M. This restriction will not, however, be imposed in this paper.

3. HOLONOMY GROUPS AND CARTAN'S THEORY

Let M be a smooth, connected, Hausdorff, paracompact manifold carrying a smooth (linear) connection Γ. For a fixed k ($1 \le k \le \infty$) and for any $p \in M$ let $C_k(p)$ denote the set of all piecewise C^k closed curves which start and end at p. If one chooses a fixed $c \in C_k(p)$, then the process of parallel transport around c using Γ can be used to map the tangent space T_pM to M at p onto itself. This map is a vector space isomorphism determined by c (and Γ) and is denoted by $f(c)$. Using the usual notation $c_1 \cdot c_2$ for the combination of two members c_1, $c_2 \in C_k(p)$ and c^{-1} for the inverse of $c \in C_k(p)$, one has $f(c_1 \cdot c_2) = f(c_1) \circ f(c_2)$ and $f(c^{-1}) = (f(c))^{-1}$. The set of all such isomorphisms $f(c)$ arising from all members $c \in C_k(p)$ is thus a subgroup of the group $GL(T_pM)$ of all isomorphisms of T_pM. This subgroup, which turns out to be independent of the value of k ($1 \le k \le \infty$), is called the *holonomy group of* M (*with respect to* Γ) *at* p or the *holonomy group* of Γ at p or simply the *holonomy group of* M *at* p if Γ is understood (Kobayashi and Nomizu, 1963/1969, Vol. I). Now since M is assumed connected, it follows that M is necessarily path connected and then the holonomy groups at any two points of M are isomorphic. One thus speaks of the *holonomy group of* M (*or* Γ, *etc.*) and denotes it by Φ. If the above operation is repeated, but this time only using members of $C_k(p)$ which are homotopic to zero, one similarly arrives at the *restricted holonomy group of* M (or Γ, etc.), which is denoted by Φ^0 (and if M is simply connected it follows that $\Phi = \Phi^0$). It turns out that Φ is a Lie group and in fact a Lie subgroup of $GL(T_pM)$ and that Φ^0 is the connected component of the identity of Φ.

Any nontrivial subspace V of T_pM which is mapped onto itself by all members of the holonomy group of M at p is called a *holonomy invariant subspace.* If one parallel transports the members of a holonomy invariant subspace V to each point of M, one obtains an m-dimensional (holonomy invariant in an obvious sense) distribution on M, where $m = \dim V$, which is smooth because Γ is smooth. If Γ is *symmetric*, this distribution is necessarily *integrable* and its maximal integrable submanifolds (which are holonomy invariant in an obvious sense) are *totally geodesic* and *autoparallel.* If such a subspace V exists, then Γ is known as *reducible.*

Now return to Cartan's theory and consider the holonomy group of the standard Cartan connection Γ on $\mathbf{R}^4$ considered in the previous section. Using the original coordinates described in that section, choose a basis $(\partial/\partial x^a)_p$ for T_pM. The holonomy group of Γ regarded as a group of isomorphisms $T_pM \mapsto T_pM$ is represented in this basis by a connected Lie subgroup H of the Lie group $GL(4, \mathbf{R})$ of all 4×4 nonsingular matrices. Now the conditions that the vector fields $(\partial/\partial x^\alpha)$ and the covector field (dx^0) are covariantly constant mean that they are preserved under parallel transport around any closed curve based at p. From this it follows that H is a connected Lie subgroup of the three-dimensional Lie group of matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{bmatrix}, \qquad a, b, c \in \mathbf{R} \tag{6}$$

The subgroup represented by the matrices (6) is in fact Lie isomorphic to the (Abelian) Lie group ($\mathbf{R}^3$, +) under the isomorphism which associates the matrix in (6) with $(a, b, c) \in \mathbf{R}^3$.

The Lie algebra of the Lie group represented by (6) is the vector space of all matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{bmatrix}, \qquad \alpha, \beta, \gamma \in \mathbf{R} \tag{7}$$

with the binary operation of matrix commutation. Any matrices A and B belonging to the set (7) satisfy $AB = 0$ and so any such matrix is *nilpotent*. The matrices in (6) can thus be easily obtained from those in (7) by exponentiation since $\exp A = I + A$. [The Lie algebra (7) can also be derived, somewhat indirectly, by using expressions given earlier for $R^a{}_{bcd}$ and Γ^a_{bc} to construct the *infinitesimal holonomy algebra* (Kobayashi and Nomizu, 1963/1969, Vol. I) and then using the fact that this Lie algebra is a Lie subalgebra of the holonomy algebra.] In fact the matrices in (7) are of the form $A^a{}_b = \psi^a t_b$, where $\psi^a = (0, \alpha, \beta, \gamma$ and $t_a = (1, 0, 0, 0)$. In general [that is, when the functions F^α in (1) are not chosen in a special way—in a sense easily made precise] the holonomy group H is three-dimensional. That is to say, H is the Lie group represented by (6) and is *noncompact* and also *exponential* (i.e., each member is the exponential of some member of its Lie algebra).

Any member of H is represented here by a matrix A of the form (6) and can be written as

$$A^a{}_b = \delta^a{}_b + \chi^a t_b$$

with $\chi^a = (0, a, b, c)$ and t_a as above. All eigenvalues of A equal unity and its Jordan form (Segre type) is $\{(211)\}$. The three corresponding independent eigendirections span the subspace of T_pM tangent to the space slice at p and it follows that the space slices themselves are *holonomy invariant*. For each such matrix A χ^a is uniquely determined by A up to a scaling and represents the eigendirection of A corresponding to the nonsimple elementary divisor.

4. LOCAL CONSIDERATIONS

The standard Cartan connection described so far is, by definition, a global structure on $\mathbf{R}^4$ and defined in a global chart for $\mathbf{R}^4$. From the mathematical viewpoint it might be asked how one could in some sense extend the definition to any manifold. However, from the physical viewpoint it would be appropriate to retain the idea of the standard Cartan connection *locally.* To this end the following definitions are suggested. They will be given for a four-dimensional manifold, but are clearly easily modified to suit any dimension.

Let M be a four-dimensional (smooth, connected, Hausdorff, paracompact) manifold. A smooth symmetric connection Γ on M is called *locally (standard) Cartan* if for each $p \in M$ there exists a coordinate domain containing p in which the only nonvanishing coefficients of Γ are of the form Γ^{α}_{00}. Thus, in M (and with the appropriate interpretations on the coordinates) one locally reproduces Cartan's geometrization of mechanics. Again let Γ be a smooth symmetric connection on M. Then Γ will then be called a *general Cartan connection for M* if M admits three pointwise independent smooth, covariantly constant global vector fields X, Y, and Z and a smooth, global covariantly constant 1-form τ which annihilates X, Y, and Z [i.e., $X(\tau) = Y(\tau) = Z(\tau) = 0$]. It follows that the smooth type (2, 0) tensor field h given by

$$h = X \otimes X + Y \otimes Y + Z \otimes Z$$

is everywhere rank 3, covariantly constant, and is annihilated by τ ($h^{ab}\tau_b = 0$). The two structures are closely related to each other and to the holonomy group structure described in the previous section, as the following theorem shows.

Theorem 1. Let M be a four-dimensional (smooth, Hausdorff, connected, paracompact) manifold with a smooth symmetric connection Γ and consider the following conditions on Γ.

(i) Γ is a general Cartan connection for M.
(ii) Γ is a local Cartan connection for M.
(iii) If $p \in M$, there exists a basis of T_pM with respect to which the holonomy group of Γ is represented by matrices of the form (6)

It then follows that (i) $\Leftrightarrow$ (iii), (i) $\Rightarrow$ (ii), and if M is simply connected, then (i), (ii), and (iii) are equivalent.

Proof. The fact that (i) and (iii) are equivalent follows immediately since vectors (respectively covectors) which are invariant under the holonomy give rise to global covariantly constant vector (respectively covector) fields on M

by parallel translation, and conversely. Now suppose that (i) holds. The vector fields X, Y, and Z which are generated by condition (i) satisfy

$$[X, Y] = [Y, Z] = [X, Z] = 0$$

and thus a coordinate system x^a about any point $p \in M$ can be chosen so that

$$X = \frac{\partial}{\partial x^1} \qquad Y = \frac{\partial}{\partial x^2} \qquad Z = \frac{\partial}{\partial x^3}$$

Also the covariantly constant 1-form τ is necessarily locally a gradient and so, by reducing the coordinate domain if necessary, it can be written as $\tau = \chi_{,a}$, where, because of the annihilation conditions, $\chi = \chi(x^0)$. In the new coordinate system (χ, x^α) the vector fields X, Y, and Z retain their above components and $\tau = dx^0$. It follows that equation (1) holds in this new coordinate system and so (ii) holds. The final part of the theorem will be established if, with the simply connected assumption, it can be shown that (ii) $\Rightarrow$ (i). This can be done using an argument very similar to that used in Hall (1989).

5. CARTAN CONNECTION ON A REDUCED BUNDLE

Consider again the standard Cartan connection on $M = \mathbf{R}^4$. Intuitively one has given the absolute time function x^0 and the space slices of constant x^0. Suppose one introduces 'rigid' Euclidean coordinates x^α in each slice giving a coordinatization $x^a = (x^0, x^\alpha)$ of $\mathbf{R}^4$. In this coordinate system the connection is imposed by the equation (1) for its coefficients. Any other such coordinate system x'^a is related to the one above by $x'^0 = x^0$, $x'^\alpha = A^\alpha{}_\beta x^\beta + f^\alpha(x^0)$, where A is a constant 3×3 orthogonal matrix [the constancy being enforced by the necessity of preserving the form (1) for the connection coefficients in the system x'^α]. The geodesic equations in the systems x^a and x'^a yield equations like (4) and (5) and show that the 'force' is not a well-defined quantity on M (Misner *et al.*, 1973). Alternatively, one could regard this ambiguity as one in the initial choice of inertial frame. If one regards the first frame above as inertial, then, if the second is also inertial, $\ddot{f}^\alpha = 0$ and the force is then defined by the original choice of inertial frame. However, *any* such frame described above could initially be regarded as inertial. It should be noted for later reference that the ambiguity in the force at any $p \in M$ is a member $\ddot{f}^\alpha(p)$ of $\mathbf{R}^3$ and that the vector field on M with components $(1, 0, 0, 0)$ everywhere in the system x^a (which can be regarded as everywhere tangent to the world lines of the 'observer' associated with this coordinate system) has components $(1, \dot{f}^\alpha(p))$ at $p \in M$ in the system x'^a.

One can now describe this phenomenon in a more precise way by recalling that the holonomy group of the standard Cartan connection Γ on

$\mathbf{R}^4$ is, in the general case, the Lie group ($\mathbf{R}^3$, +). Now Γ is a linear connection on the frame bundle $L(M)$ of M and, according to the holonomy reduction theorem (Kobayashi and Nomizu, 1963/1969, Vol. I), it can be reduced to a connection on a principal fiber bundle over M with structure group ($\mathbf{R}^3$, +), denoted by $C(M, \mathbf{R}^3)$. This reduction can be visualized as applying to the subbundle of frames which at each $p \in M$ consists of the values at p of the vector fields $\partial/\partial x^\alpha$ in the original coordinate system above together with any of the three-parameter family of vectors at p with components $(1, \alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \mathbf{R}$. A (local or global) choice of this latter vector T at each $p \in M$, which gives a vector field on M with components $(1, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$, with $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ functions of x^0 only, fixes a 'gauge' and can be interpreted as a choice of observer (i.e., a choice of one of the above coordinate systems up to a time-independent translation in each space slice). The connection 1-forms of the reduced connection (gauge potentials) are $\mathbf{R}^3$-valued and can be shown to give the components of the force in a coordinate system corresponding to T. A change of gauge (i.e., a change of vector field corresponding to a choice of α, β, γ) corresponds to a change of observer and the associated change in the gauge potential corresponds to the 'acceleration transformations' given by equation (5). The geometrical status of F^α thus becomes clearer after the introduction of the reduced connection since the transformation law obeyed by F^α is in fact the transformation law of an appropriate connection under a gauge change. More details regarding the construction of the reduced subbundle and associated connection can be found in Haddow (1993).

6. SOME EXTENSIONS OF CARTAN'S METHOD

6.1. Lagrangian Mechanics

Suppose that one has an N-particle system with holonomic constraints in classical (Lagrangian) mechanics. One can then describe it in a well known way as an unconstrained system on configuration space M'. Here M' is a smooth n-dimensional connected manifold with a global smooth positive-definite metric g. The motion of the system in configuration space is then given by a smooth path c in M' which, in local coordinates in M', satisfies

$$\frac{d^2x^\alpha}{dx^{0^2}} + \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} \frac{dx^\beta}{dx^0}\frac{dx^\gamma}{dx^0} + Q^\alpha = 0 \tag{8}$$

where Greek indices now run from 1 to n (and Latin indices from 0 to n). The Q^α are the generalized force vector components, $\left\{{\alpha \atop \beta\gamma}\right\}$ are the usual (Levi-Civita) Christoffel symbols constructed from the metric g, and x^0 is Newtonian absolute time.

Construct the connected product manifold $M = \mathbf{R} \times M'$ and build a smooth symmetric connection Γ on M in the following way. Let U be any chart domain of M' and define the coefficients of Γ in the natural coordinate system for $\mathbf{R} \times U \subset M$ by

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}, \qquad \Gamma^{\alpha}_{00} = Q^{\alpha} \tag{9}$$

with all other connection components zero. It is easy to check that if U' is another chart domain of M' with $U' \cap U \neq \varnothing$, then the corresponding coefficients of Γ in $\mathbf{R} \times U'$ are related to those in $\mathbf{R} \times U$ in the correct way. Thus by employing definition (9) over an atlas of such charts of M, one sees that Γ is a smooth symmetric connection for M.

One can now evaluate some of the properties of the connection Γ on M. First, by using the above atlas of chart domains of M, one can show that the local 1-forms defined in each member of this atlas by $\tau = dx^0$ give rise to a global smooth 1-form τ on M. This 1-form is clearly a global gradient, $\tau = df$, where $f: M \mapsto \mathbf{R}$ is the obvious smooth function whose level surfaces are submanifolds of M diffeomorphic to M', and it annihilates all vector fields which are tangent to M'. It then follows from (9) that τ is covariantly constant with respect to the connection Γ.

Second, again by using the above atlas of M, the local second-order symmetric tensors with components

$$h^{ab} = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & g^{\alpha\beta} \end{bmatrix} \tag{10}$$

in each such coordinate system, where $g^{\alpha\beta}$ are the (raised) components of g in the associated coordinate chart for M', give rise to a global smooth type (2, 0) tensor h on M which everywhere has rank n. It then follows from (9) that *h is covariantly constant.*

Third, the holonomy group of Γ can be calculated from the previous results. For any point $p \in M$ it is possible to choose a coordinate system whose domain contains p and such that $h(p)$ takes its Sylvester canonical form $\alpha = \text{diag}(0, 1, \dots, 1)$. Clearly the holonomy group is then a subgroup of the Lie group of matrices

$$\{A \in GL(n + 1, \mathbf{R}) | A\alpha A^T = \alpha\}$$

Any member of this group can be written in the form

$$\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{v} & P \end{bmatrix} \tag{11}$$

where $\mathbf{v}$ represents a column vector in $\mathbf{R}^n$ and P is an $n \times n$ orthogonal matrix (i.e., $PP^T = I$). It follows from (11) that the submanifolds M' in M (using an obvious abuse of notation) are holonomy invariant and hence totally geodesic and autoparallel and that the symmetric connection naturally induced in them by Γ (Kobayashi and Nomizu, 1963/1969, Vol. II) is the Levi-Civita connection arising from g. It is remarked here that the group of all matrices (under matrix multiplication) of the form (11) is the semidirect product $\mathbf{R}^n \times_f O(n)$, where f is the homomorphism from the group $O(n)$ to the automorphism group of $\mathbf{R}^n$ which associates $P \in O(n)$ with the automorphism $\mathbf{v} \mapsto P\mathbf{v}$ ($\mathbf{v} \in \mathbf{R}^n$).

Fourth, and again using the coordinate chart (x^a, $\mathbf{R} \times U$) as above and identifying x^0 with Newtonian absolute time t, the controlling equation for particle paths (8) is (together with the equation $d^2x^0/d\lambda^2 = 0$) *the geodesic equation in M with respect to Γ with λ as an affine parameter.*

Finally, one can determine under which conditions Γ is a metric connection, at least in the 'generic' case when M' with its Levi-Civita connection admits no local or global covariantly constant vector fields. Suppose Γ is compatible with a metric q of any signature on M. It follows that M admits a global nowhere-zero covariantly constant vector field $\tilde{\tau}$ with components $\tilde{\tau}^a = q^{ab}\tau_b$. This vector field is, by construction, everywhere q-orthogonal to the submanifolds M'. Also, it cannot be q-null at any point of M, because otherwise it would be q-null everywhere on M and hence everywhere tangent to the submanifolds M'. Then, since the Levi-Civita connection of g in each M' is induced by Γ, the vector field $\tilde{\tau}$ would give rise to a vector field in each M' which is covariantly constant with respect to this Levi-Civita connection, contradicting the generic condition on its holonomy group. Even if the generic condition on the holonomy group of the connection on M' is dropped, then information is still available (Haddow, 1993). It follows that $\tilde{\tau}$ is nowhere g-null on M. Hence M is locally decomposable and admits local coordinates x'^a about any point adapted to the decomposition in which $\tilde{\tau}$ and τ are given by $\tilde{\tau} = \pm\partial/\partial x'^0$ and $\tau = dx'^0$ (depending on the signature of q). Since $\tau = dx^0$ in the original (natural) coordinate system, the transformation between the original coordinates x^a and the new coordinates x'^a satisfies $x'^0 = x^0$ (after setting an arbitrary constant to zero). In the original coordinates, $\tilde{\tau}$ then has components $\tilde{\tau}^a$ satisfying $\tilde{\tau}^0 = \pm 1$ and since $\tilde{\tau}$ is covariantly constant with respect to Γ and $\Gamma^\alpha_{0\beta} = 0$, one has

$$\tilde{\tau}^\alpha{}_{,\beta} + \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \tilde{\tau}^\gamma = 0 \tag{12}$$

where $\{{}^{\alpha}_{\beta\gamma}\}$ are associated with g. This implies that either $\tilde{\tau}^\alpha \equiv 0$ or the original manifold M' admits a nowhere-zero local covariantly constant vector field. The latter is ruled out by the generic condition on the Levi-Civita connection on M' and so $\tilde{\tau} = \pm\partial/\partial x^0$ [and so these two coordinate systems are linked by $x'^0 = x^0$, $x'^\alpha = x'^\alpha(x^\beta)$]. The covariant constancy of $\tilde{\tau}$ together with the above values for its components in the original coordinate system now give Γ^α_{00} $(= Q^\alpha) = 0$ and in this sense the original physical system was 'trivial.' Conversely, if $\Gamma^\alpha_{00} = 0$ holds in addition to (9) in the original coordinate system, then the local vector field $\tilde{\tau}$, defined in each such coordinate system by $\tilde{\tau} = \pm\partial/\partial x^0$, is covariantly constant and the tensor components $h^{ab} \pm \tilde{\tau}^a\tilde{\tau}^b$ in each of these coordinate systems then give rise to a *global* metric on M which is compatible with Γ. *Thus* Γ *is a metric connection if and only if the original physical system was trivial in the above sense.*

If Γ is a metric connection on M, then the fact that the vector field $\tilde{\tau}$ is covariantly constant means that $\tilde{\tau}(p)$ is preserved under the holonomy group of Γ at p and the holonomy matrices in the natural coordinate systems take the form (11) with $\mathbf{v} = 0$. The holonomy group is then Lie isomorphic to a Lie subgroup of some (pseudo-) orthogonal group on $\mathbf{R}^{n+1}$, as it should be. In this case the connection Γ is not necessarily flat (as in the standard Cartan case), because of the curvature 'residing' in the connection on M'.

6.2. Electromagnetic Theory

Consider a particle P with electric charge e moving in an electromagnetic field represented by the electric and magnetic 3-vectors $\mathbf{E}$ and $\mathbf{B}$. Then, to the usual approximation, the force on P when it has velocity $\mathbf{v}$ in some inertial frame is given by the Lorentz force law as $\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. With units chosen so that the charge to mass ratio of P is unity, the Newtonian equations of motion for P in the usual Cartesian coordinates can be written as

$$\frac{d^2x^\alpha}{dx^{0^2}} - E^\alpha - B^\alpha{}_\beta \frac{dx^\beta}{dx^0} = 0 \tag{13}$$

Newtonian absolute time is denoted by x^0, B is the skew matrix

$$B^\alpha{}_\beta = \begin{bmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{bmatrix} \tag{14}$$

and the convention for Greek and Latin indices is as in Section 2. Now regard the space and absolute time coordinates here as coordinates on $\mathbf{R}^4$ $(= M)$ with $\mathbf{E}$ and $\mathbf{B}$ regarded as smooth functions of x^α and x^0 and introduce into $\mathbf{R}^4$ a smooth symmetric connection Γ whose only nonvanishing coefficients in this coordinate system are given by

$$\Gamma^{\alpha}_{00} = -E^{\alpha}, \qquad \Gamma^{\alpha}_{\beta 0} = \Gamma^{\alpha}_{0\beta} = -\frac{1}{2} B^{\alpha}{}_{\beta} \tag{15}$$

Then by arguments similar to those given in Section 2 it follows that each space slice is a totally geodesic and autoparallel submanifold of M and that one can introduce a 1-form $t = dx^0$ and a tensor h with components $h^{ab} = \text{diag}(0, 1, 1, 1)$ which are easily shown to be covariantly constant with respect to Γ. Also t annihilates h. Next, any affinely parametrized geodesic $x^a(\lambda)$ in M which is not tangent to a space slice at its initial point (and hence is never tangent to a space slice) has equations

$$\frac{d^2x^\alpha}{d\lambda^2} - E^\alpha\left(\frac{dx^0}{d\lambda}\right)^2 - B^\alpha{}_\beta \frac{dx^\beta}{d\lambda}\frac{dx^0}{d\lambda} = 0, \qquad \frac{d^2x^0}{d\lambda^2} = 0 \tag{16}$$

which is just the dynamical equation (13) above.

The holonomy group of Γ can be calculated at p in M using the basis $(\partial/\partial x^a)_p$ and is easily found to consist of matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & & & \\ b & & P & \\ c & & & \end{bmatrix} \tag{17}$$

where a, b, $c \in \mathbf{R}$ and $P \in SO(3)$. The complete set of matrices (17) constitutes the semi-direct product $\mathbf{R}^3 \times_f SO(3)$, where f is the homomorphism associating $P \in SO(3)$ with the $\mathbf{R}^3$ automorphism $\mathbf{v} \mapsto P\mathbf{v}$.

Since each space slice is an autoparallel submanifold of M, the symmetric connection Γ on M defined above naturally induces a symmetric connection in each space slice. Further, if ∇ denotes covariant differentiation with respect to Γ and ∇' denotes covariant differentiation with respect to the induced connection in a particular space slice, then for vector fields A and B which are everywhere tangent to the space slice (and hence induce vector fields A' and B' in the space slice in a natural way) $\nabla_A B$ is tangent to the space slice and induces the vector field $\nabla'_{A'} B'$ in it. Now it immediately follows from equation (15) defining Γ that $\nabla_{X_i} X_j = 0$, where $X_k = \partial/\partial x^k$ and $k = 1, 2, 3$. Hence $\nabla'_{X'_i} X'_j = 0$ and as a consequence *the connection* ∇' *on the space slice is flat* (and, in fact, is the Levi-Civita connection for the metric on the space slice naturally induced by h). However, the connection Γ on M may, but does not necessarily, admit any (local or global) covariantly constant vector fields. In fact the condition that a vector field W on M is covariantly constant is, in the above coordinate system on M, equivalent to the conditions

$$W^{\alpha}{}_{,\beta} - \frac{1}{2} B^{\alpha}{}_{\beta} W^0 = 0 \tag{18}$$

$$W^{\alpha}{}_{,0} - E^{\alpha} W^0 - \frac{1}{2} B^{\alpha}{}_{\beta} W^{\beta} = 0 \tag{19}$$

$$W^0{}_{,a} = 0 \tag{20}$$

Now (20) is equivalent to W^0 being constant and so after a constant scaling of W one need only consider the cases $W^0 = 0$ and $W^0 = 1$. A consideration of the (source-free) electromagnetic field represented by the plane wave whose only nonvanishing electric and magnetic field components are

$$B_3 = B_0 \sin(kx^1 - \omega x^0), \qquad E_2 = cB_0 \sin(kx^1 - \omega x^0) \tag{21}$$

where B_0, c, ω, and k are nonzero constants, shows that the associated connection Γ on M admits just a single independent covariantly constant vector field W (and that this comes from the case $W^0 = 0$) given by $W = \partial/\partial x^3$. However, if one linearly superimposes upon the field (21) the electromagnetic field whose only nonvanishing components are

$$B_1 = B_0 \sin(kx^2 - \omega x^0), \qquad E_3 = cB_0 \sin(kx^2 - \omega x^0) \tag{22}$$

then the connection associated with this combined field can be seen, using equations (18)–(20), to admit no (local or global) covariantly constant vector fields.

The final remark of the previous paragraph shows that the connection Γ is not necessarily flat. It is also not necessarily a metric connection. In fact Γ *is metric if and only if it is flat.* To see this, one notes as in the case of the standard Cartan connection, since $M = \mathbf{R}^4$ is simply connected, that if Γ is flat, then it is metric. Conversely, suppose that Γ is a metric connection with compatible metric g of arbitrary signature and corresponding curvature tensor $R^a{}_{bcd}$. The above argument regarding the inducing of Γ onto the space slices, together with the Ricci identity, shows that, at any point $p \in M$, $R^a{}_{bcd}A^bB^cC^d = 0$ whenever A, B, and C are vector fields on M tangent to the space slice at p. Thus for *any* such tangent vectors B and C at p the tensor $F^a{}_b = R^a{}_{bcd}B^cC^d$ at p, which is skew-self-adjoint with respect to g (i.e., $g_{ea}F^e{}_b + g_{be}F^e{}_a = 0$), satisfies $F^a{}_bA^b = 0$ for all $A \in T_pM$ tangent to the space slice at p and hence has rank less than or equal to one. Since such a skew-self-adjoint tensor must have even rank, it follows that all such tensors $F^a{}_b$ are zero at all points of M. As a consequence the Riemann tensor components R_{abcd} ($= g_{ae}R^e{}_{bcd}$) are, at each $p \in M$, sums of terms of the form $G_{ab}H_{cd}$, where G and H are drawn from a set of three linearly independent skew tensors $t_{[a}p^{(i)}_{b]}$ ($i = 1, 2, 3$) (square brackets denote the usual skew-symmetrization), where each $p_a^{(i)}$ is not parallel to t_a. Now the fact that h is covariantly constant together with the Ricci identity gives

$$h^{be}R^a{}_{ecd} + h^{ae}R^b{}_{ecd} = 0 \tag{23}$$

Since the above tensors $t_{[a}p_{b]}^{(i)}$ are linearly independent, no linear combination of the $p_a^{(i)}$ can be proportional to t_a and hence no such linear combination can annihilate h (since t is unique up to scaling with respect to this last property). It finally follows by substituting this information into (23) that the Riemann tensor is zero at each $p \in M$ and so Γ is flat.

7. DEGENERATE METRIC CONNECTIONS

The generalizations of Cartan's idea above have led naturally to a study of a smooth n-dimensional manifold M admitting a smooth symmetric connection Γ and a smooth second-order symmetric type (2, 0) tensor h which is covariantly constant with respect to Γ and of rank $n - 1$ everywhere. This section will explore some of the existence and uniqueness problems associated with structures like h and Γ and extends work in Hall and Haddow (1994), but first some definitions are required. Throughout this section M will be a connected, paracompact, n-dimensional, smooth Hausdorff manifold and all structures discussed will be assumed smooth. If M admits a global smooth symmetric type (2, 0) tensor h of rank $n - 1$ at each $p \in M$ (henceforth called a *degenerate metric* for M), then two natural distributions arise on M. The first is the *kernel distribution* K^* of h, which associates each point $p \in M$ with the unique one-dimensional subspace of the cotangent space T_p^*M to M at p which annihilates h. The second is the *h-distribution* K, which associates with each $p \in M$ the unique $(n - 1)$-dimensional subspace of T_pM which is annihilated by each member of K^* at p. Thus

$$K^*(p) = \{\omega \in T_p^*M \,|\, (h(p))(\omega) = 0\}$$

$$K(p) = \{v \in T_pM \,|\, \omega(v) = 0 \; \forall \omega \in K^*(p)\}$$

for each $p \in M$. Here $h(p)$ is regarded as a linear map $T_p^*M \mapsto T_pM$ in the usual way. It follows from the smoothness of h that these two distributions are smooth in the sense that K^* can be locally spanned by a (local) smooth 1-form and K by $(n - 1)$ (local) smooth vector fields (see, e.g., Hall and Rendall, 1989).

Since M is paracompact, it necessarily admits a linear connection and if M admits a degenerate metric h and a connection Γ with respect to which h is covariantly constant, then Γ will be said to *fit* h and will be called a *degenerate metric connection*. Under these circumstances the distributions K^* and K are easily seen to be invariant under the holonomy group of Γ.

Given such a tensor field h on M, the distributions K^* and K are called *integrable* if, in the first case, K^* can be locally spanned by a local exact (i.e., gradient) 1-form and, in the second case, K can be locally spanned by a (local) involutive system of $(n - 1)$ smooth vector fields. It follows from a standard result (Spivak, 1970) that K^* is integrable if and only if K is. It is, however, possible that neither is integrable. To see this, consider the three vector fields on $\mathbf{R}^4$ given by (Brickell and Clark, 1970)

$$X = \frac{\partial}{\partial x} \qquad Y = \frac{\partial}{\partial y} \qquad Z = \frac{\partial}{\partial z} + e^x \frac{\partial}{\partial t} \tag{24}$$

where (x, y, z, t) is the standard global chart for $\mathbf{R}^4$. This system is not involutive (since $[X, Z] = e^x(\partial/\partial t)$ does not lie in the associated distribution), but is, nevertheless, the h-distribution of the degenerate metric given by

$$h = X \otimes X + Y \otimes Y + Z \otimes Z$$

But if h is a degenerate metric and Γ is a *symmetric* connection for M and if Γ fits h, then the distributions K^* and K are invariant under the holonomy group arising from Γ and are hence integrable (Kobayashi and Nomizu, 1963/1969, Vol. I). However, it should be stressed here that integrability of the distributions is a consequence of the fact that Γ is *symmetric* and may fail if Γ is not symmetric. An example of this behavior will be given later.

Suppose now that M admits a degenerate metric h. Under what conditions does M admit a connection or a symmetric connection which fits h and what can be said about the uniqueness of such connections? The question contains the analog of the Levi-Civita problem for (nondegenerate) metrics on M. Clearly from the remarks in the previous paragraph, the associated kernel and h-distributions must be integrable in order for a *symmetric* connection to exist and fit h. This turns out to be the only restriction required, as the next theorem shows.

Theorem 2. Let M be a connected, paracompact, smooth, Hausdorff, n-dimensional manifold which admits a degenerate metric h. It then follows that:

(i) M admits a smooth connection which fits h.
(ii) M admits a smooth symmetric connection which fits h if and only if the kernel distribution K^* of h (equivalently the h-distribution K) is integrable.
(iii) The connection (or symmetric connection if one exists) which fits h is not unique.

Proof. Since M is paracompact, it admits a smooth, global, *positive-definite* metric γ (see, e.g., Kobayashi and Nomizu, 1963/1969, Vol. I). Also, if $p \in M$, there exists an open neighborhood U of p and a smooth (nowhere zero) 1-form ρ on U which spans K^* at each point of U. Regarding γ in the usual way as an isomorphism $T_qM \leftrightarrow T_q^*M$ at each $q \in M$, one then has a local smooth, nowhere-zero vector field $T(p) = \gamma^{-1}(\rho(p))$ on U and T may (and will) be regarded as a γ-unit vector field on U by rescaling if necessary. By repeating this construction over an open cover of such neighborhoods U one sees that at any $p \in M$ the corresponding vectors T arising at $p \in M$ agree up to a sign. Hence a global type (2, 0) smooth symmetric tensor field H is defined on M which is given unambiguously in any such neighborhood U by $T \otimes T$. Now let $\gamma' = h + H$. The matrix $\gamma'(p)$ is nonsingular for any $p \in M$ because the equation $(h^{ab} + T^aT^b)\omega_b = 0$ at p is easily shown, after a contraction with ρ_a, to yield the contradiction $T^a\rho_a = 0$ at p. The tensor γ' is thus seen to be a global smooth (contravariant) metric for M and hence has a corresponding (symmetric) Levi-Civita connection Γ'. Now define in any of the above coordinate domains the functions Γ^a_{bc} (Walker, 1955)

$$\Gamma^a_{bc} = \Gamma'^a_{bc} + T^a(T_{b|c} + T_{c|d}T^dT_b) \tag{25}$$

where a vertical stroke denotes covariant differentiation with respect to Γ' and $T_a = \gamma'_{ab}T^b$. Then by the above construction these functions are the local coefficients of a (not necessarily symmetric) connection Γ on M. Again in this coordinate domain, and using a semicolon to denote differentiation with respect to Γ, one finds after a short calculation that

$$h^{ab}{}_{;c} = h^{ab}{}_{|c} + h^{ad}P^b{}_{dc} + h^{db}P^a{}_{dc} = 0 \tag{26}$$

The tensor $P^a{}_{bc}$ is defined by $P^a{}_{bc} = \Gamma^a_{bc} - \Gamma'^a_{bc}$ and the facts that $\gamma'^{ab}{}_{|c} = 0$ and $T_a \propto \rho_a$ ($\Rightarrow h^{ab}T_b = 0$) and $\gamma'^{ab}T_b = T^a$ [$\Rightarrow (T^aT_a)_{|b} = 0 \Rightarrow T_{a|b}T^a = 0$] have been used. This completes the proof of part (i). For part (ii) it has already been established that if such a symmetric connection exists, then integrability follows. Now suppose that K^* is integrable. Then in each of the above coordinate domains T_a ($\propto\rho_a$) satisfies the well-known restriction $T_{[a|b}T_{c]} = 0$ which follows since the (Levi-Civita) connection Γ' is symmetric. When this relation is written out and contracted with T^b (and use is made of the above result $T_{a|b}T^a = 0$) one finds

$$T_{[c}T_{a]|b}T^b + T_{[c|a]} = 0 \tag{27}$$

It then follows from (27) that $\Gamma^a_{[bc]} = 0$ and so Γ is symmetric. To establish (iii), choose (as one always can under the conditions of the theorem) a global smooth (not identically zero) vector field k on M. In the above charts the coefficients

$$\bar{\Gamma}^a_{bc} = \Gamma^a_{bc} + k^aT_bT_c$$

give rise to a well-defined smooth connection $\bar{\Gamma}$ on M. This connection is distinct from Γ, fits h, and is symmetric if and only if Γ is symmetric. This completes the proof.

It is remarked that by utilizing a degenerate metric such as, for example, was constructed from the vector fields in (24), one can use (i) above to construct (nonsymmetric) connections whose holonomy groups are reducible but where holonomy invariant subspaces at each $p \in M$ do not give rise to integrable submanifolds. Also (iii) should be compared and contrasted with the guaranteed uniqueness of the (symmetric) Levi-Civita connection of a (nondegenerate) metric on M.

If h is a degenerate metric for M and Γ a connection on M which fits h, then, as remarked earlier, the kernel distribution K^* is holonomy invariant. It follows that the local 1-forms ρ spanning K^* are *recurrent*; $\rho_{a;b} = \rho_a q_b$, where a semicolon denotes covariant differentiation with respect to Γ and q is another smooth local 1-form [a further study of the *recurrence 1-form* q can be found in Haddow (1993) and Hall (1991)]. It may not be possible to locally scale these local recurrent 1-forms in order that they are covariantly constant. This would be the case if and only if each such q was locally exact [if $q_a = \psi_{,a}$ for some local function ψ, then $(e^{-\psi}\rho_a)_{;b} = 0$]. For example, the standard Cartan connection in Section 2 admits a global covariantly constant 1-form dx^0 which spans the kernel of h. But if one modifies this connection by introducing one extra nonzero connection coefficient Γ^0_{00} which is not just a function of x^0 then one still has the degenerate metric h, but the kernel distribution cannot now be spanned by a local covariantly constant 1-form in some neighborhood of each $p \in M$.

These features are closely related to certain properties of the holonomy group of the degenerate metric connection Γ, special cases of which have already been mentioned in earlier examples. Let M, Γ, and h be as above and note that since h is covariantly constant it has the same representative (ordered) *Sylvester* matrix at each $p \in M$, say

$$\beta = \mathrm{diag}(0, 1, \ldots, 1, -1, \ldots, -1)$$

with r positive and s negative entries ($r + s + 1 = n$). It is then clear that the holonomy group of Γ is isomorphic to a Lie subgroup of the Lie group $DM(r, s)$ [degenerate metric group of signature (r, s)] defined by

$$DM(r, s) = \{A \in GL(n, \mathbf{R}) | A\beta A^T = \beta\} \tag{28}$$

By explicitly writing out the defining condition in a Sylvester basis of T_pM at some $p \in M$ it follows that any $A \in DM(r, s)$ can be written as

$$A = \begin{bmatrix} d & \mathbf{0}^T \\ \mathbf{v} & P \end{bmatrix} \tag{29}$$

In (29), $d \in \mathbf{R}$ ($d \neq 0$), $\mathbf{v} \in \mathbf{R}^{n-1}$, and the $(n - 1) \times (n - 1)$ matrix $P \in$

$O(r, s)$. In the corresponding dual basis at p the 1-form ρ with components $(1, 0, \ldots, 0)$ spans K^* and, under parallel transport around a closed curve at p whose associated transformation of T_pM is represented by the holonomy matrix (29), is scaled by the factor d. One thus obtains the following result.

Theorem 3. Let M be a connected, paracompact, smooth, Hausdorff, n-dimensional manifold which admits a degenerate metric h of signature (r, s) and connection Γ which fits h. The holonomy group is then isomorphic to some subgroup of $DM(r, s)$ and K^* is locally spanned by a recurrent 1-form. Further, and using the above conventions, $d = 1$ for all members of the holonomy group if and only if K^* can be spanned by a global covariantly constant 1-form.

In the last part of the above theorem (i.e., when $d = 1$) the holonomy group is thus a subgroup of the group of all matrices of the form (29) with $d = 1$. The latter group is the semi-direct product $\mathbf{R}^{n-1} \times_f O(r, s)$, where f is the homomorphism from the group $O(r, s)$ to the automorphism group of $\mathbf{R}^{n-1}$ which associates $P \in O(r, s)$ with the automorphism $\mathbf{v} \mapsto P\mathbf{v}$ ($v \in \mathbf{R}^{n-1}$). This semi-direct product is the *pseudo-Euclidean group of signature* (r, s).

As a final remark, another example of a degenerate metric connection can be described. Let M be an n-dimensional smooth manifold with smooth *Lorentz* metric g. Let N be a *null* submanifold of M so that at each $p \in N$ there is a unique g-null direction in T_pM tangent to N. Then g induces a (covariant) degenerate metric h in N in the usual way (i.e., if i: $N \mapsto M$ is the natural inclusion, then $h = i^*g$). Further, if N is autoparallel with respect to the Levi-Civita connection in M arising from g [as, for example, occurs in the study of space-times in general relativity which possess a recurrent null vector field (Hall, 1991)], then this latter connection induces a natural connection in N which fits h.

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